# Math 429 - Exercise Sheet 10

**1.** In the previous exercise sheet, you worked out the root decompositions of the complex semisimple Lie algebras of types $A, B, C, D$. In each of those cases, write out all the roots as linear combinations of a fixed choice of simple roots.

**2.** Show that the following subset of $\mathbb{R}^n$

$$R = \Big\{ \pm e_i, \pm 2e_i \Big\}_{1 \leq i \leq n} \sqcup \Big\{ \pm e_i \pm' e_j \Big\}_{1 \leq i < j \leq n}$$

determines a **non-reduced** root system, i.e. satisfies all axioms in Definition 19, except for (136) (specifically, if $\alpha$ is a root, then we do allow $2\alpha$ or $\frac{\alpha}{2}$ to be a root). It is called "type $BC_n$".

**3.** Consider the root system of type $D_6$ with simple roots $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$, $\alpha_4 = e_4 - e_5$, $\alpha_5 = e_5 - e_6$, $\alpha_6 = e_5 + e_6$, and consider the following decomposition

$$\mathbb{R}^6 = \text{span}\Big(\alpha_6 - \varphi\alpha_1, \alpha_3 - \varphi\alpha_5, \alpha_4 - \varphi\alpha_2\Big) \oplus \text{span}\Big(\varphi\alpha_6 + \alpha_1, \varphi\alpha_3 + \alpha_5, \varphi\alpha_4 + \alpha_2\Big)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. Show that the three-dimensional subspaces in the right-hand side are orthogonal. If we let $\pi : \mathbb{R}^6 \to \mathbb{R}^3$ denote the orthogonal projection onto the second subspace, show that

$$\pi(60 \text{ roots of } D_6) = R \sqcup \varphi R$$

for some $R \subset \mathbb{R}^3$ of cardinality 30. This set $R$ determines a **non-crystallographic** root system, i.e. satisfies all axioms in Definition 19 except for (137). It is called "type $H_3$" and it is related to symmetries of a regular icosahedron.

**4.** The length of an element $w$ in the Weyl group $W$ (of a root system $R$ with a given set of simple roots $I$) is defined as

$$\ell(w) = \min\Big\{k \geq 0 \Big| \exists i_1, \dots, i_k \in I \text{ s.t. } w = s_{i_1} \dots s_{i_k}\Big\}$$

Show that $\ell : W \to \mathbb{Z}_{\geq 0}$ is a length function, i.e.

$$\ell(e) = 0$$

$$\ell(w^{-1}) = \ell(w)$$

$$\ell(w_1 w_2) \leq \ell(w_1) + \ell(w_2)$$

Show that there exists a unique element of $W$ of maximal length (how does it act on roots and on Weyl chambers?) What is the length function for the type $A_{n-1}$ root system, for which $W = S_n$?

**(*)** If $\mathfrak{g}$ is a (not necessarily semisimple) Lie algebra over a field of characteristic 0, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a Cartan subalgebra if it is nilpotent and self-normalizing:

$$\Big\{x \in \mathfrak{g} \Big| [x, \mathfrak{h}] \subseteq \mathfrak{h}\Big\} = \mathfrak{h}$$

Prove that any Cartan subalgebra is a maximal nilpotent subalgebra (the converse fails in general, though, think about the subalgebra $\mathfrak{n} \oplus \mathbb{C}I_n$ of $\mathfrak{gl}_n$).